Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 223 (2009) 438–448
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Multiple positive solutions of multi-point boundary value problem for second-order impulsive differential equations☆

Feng Meiqiang*, Xie Dongxiu

Department of Mathematics, Beijing Information Science & Technology University, Beijing, 100192, PR China

Received 15 October 2007; received in revised form 24 December 2007

Abstract

This paper is devoted to study the existence of multiple positive solutions for the second-order multi-point boundary value problem with impulse effects. The arguments are based upon fixed-point theorems in a cone. An example is worked out to demonstrate the main results.

MSC: 34B15

Keywords: Multi-point boundary value problem with impulse effects; Positive solution; Completely continuous; Fixed-point theory; Existence

1. Introduction

The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often, especially in phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. For an introduction of the basic theory of impulsive differential equations in R^n, see [1–3] and the references therein. The theory of impulsive differential equations has become an important area of investigation in recent years and is much richer than the corresponding theory of differential equations (see for instance [4–11] and their references). However, to the best of our knowledge, these papers only studied the two-point or periodic BVPs of impulsive differential equations. Being directly inspired by Ref. [14], by use of the fixed point theory in a cone, this article is devoted to study m-point BVPs for second-order impulsive differential systems.

Consider the following second-order m-point boundary value problem with impulse effects

$$\begin{cases} -x''(t) = f(t, x(t)), \quad t \in J, \quad t \neq t_k, \\ -\Delta x'|_{t=t_k} = I_k(x(t_k)), \quad k = 1, 2, \ldots, n, \\ x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i), \qquad x(1) = \sum_{i=1}^{m-2} b_i x(\xi_i). \end{cases} \tag{1.1}$$

☆ Supported by the National Natural Science Foundation of China (10771022, 10671012), the Doctoral Program Foundation of Education Ministry of China (20050007011), the Beijing Natural Science Foundation (1062005) and the Science and Technology Innovation Foundation for postgraduate students of Beijing Information Technology Institute (GA200809).
* Corresponding author.
E-mail address: fmqzxm@163.com (M. Feng).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2008.01.024

Here $J = [0, 1]$, $f \in C(J \times R^+, R^+)$, $I_k \in C(R^+, R^+)$, $R^+ = [0, +\infty)$, t_k $(k = 1, 2, \ldots, n)$ (where n is fixed positive integer) are fixed points with $0 < t_1 < t_2 < \cdots < t_k < \cdots < t_n < 1$, ξ_i $(i = 1, 2, \ldots, m-2) \in (0, 1)$ be given $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and $\xi_i \neq t_k$, $i = 1, 2, \ldots, m-2$, $k = 1, 2, \ldots, n$, $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$, where $x'(t_k^+)$ and $x'(t_k^-)$ represent the right-hand limit and left-hand limit of $x'(t)$ at $t = t_k$ respectively, $a_i, b_i \in (0, +\infty)$, $i = 1, 2, \ldots, m-2$.

For the case of $I_k = 0$, $k = 1, 2, \ldots, n$, problem (1.1) reduces to multi-point boundary value problem of ODE. The study of multi-point boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [12]. Since then a lot of attention has been paid to the study of nonlinear multi-point boundary value problems, see [12–17,19–31]. There are two papers [14,19] that are closely related to the present paper. In [14], set

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}.$$

Then $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case. If f is either superlinear or sublinear, the following multi-point boundary value problem

$$\begin{cases} u''(t) + q(t) f(u(t)) = 0, \quad 0 < t < 1, \\ u'(0) = \sum_{i=1}^{m-2} a_i u'(\xi_i), \qquad u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \end{cases} \tag{1.2}$$

was proved to have at least one positive solution, by Ma and Castaneda.

In [19], Bai and Du considered the following second-order four-point boundary value problems:

$$\begin{cases} -x'' + \lambda h(t) f(t, x(t)) = 0, \quad 0 < t < 1, \\ x(0) = a x(\xi), \qquad x(1) = b x(\eta), \end{cases} \tag{1.3}$$

where $0 < \xi < \eta < 1$, $0 \le a, b < 1$, and $h : [0, 1] \to [0, \infty)$, $f : [0, 1] \times [0, \infty) \to [0, \infty)$ are nonnegative continuous functions. By using the fixed-point index theory, Leray–Schauder degree and the upper and lower solution method, the authors established the existence, nonexistence, and multiplicity of positive solutions of BVP (1.3), where multiplicity of a positive solution means that the authors obtained two positive solutions of BVP (1.3).

So it is interesting and important to discuss the existence of positive solutions for BVP (1.1) when $I_k \neq 0$, $k = 1, 2, \ldots, n$. Many difficulties occur when we deal with them. For example, the construction of cone and operator. So we need to introduce some new tools and methods to investigate the existence of positive solutions for BVP (1.1). Moreover, the methods used in this paper are different from [14,19] and the results obtained in this paper generalize some results in [14,19] to some degree.

To obtain positive solutions of (1.1), the following fixed-point theorem in cones is fundamental which can be found in [18], pp. 93.

Lemma 1.1. *Let Ω_1 and Ω_2 be two bounded open sets in Banach space E, such that $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Let P be a cone in E and let operator $A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$ be completely continuous. Suppose that one of the following two conditions is satisfied:*

(i) *$Ax \not\geq x, \forall\, x \in P \cap \partial\Omega_1$; $Ax \not\leq x, \forall\, x \in P \cap \partial\Omega_2$.*
(ii) *$Ax \not\leq x, \forall\, x \in P \cap \partial\Omega_1$; $Ax \not\geq x, \forall\, x \in P \cap \partial\Omega_2$.*

Then, A has at least one fixed point in $P \cap (\Omega_2 \setminus \bar{\Omega}_1)$.

Remark 1. To make the reader clear what $\bar{\Omega}_2$, $\partial\Omega_2$, $\partial\Omega_1$, and $\Omega_2 \setminus \bar{\Omega}_1$ mean, we give typical examples of Ω_1 and Ω_2, e.g.,

$$\Omega_1 = \{x \in C[a, b] : \|x\|_\infty < r\}, \qquad \Omega_2 = \{x \in C[a, b] : \|x\|_\infty < R\}$$

with $0 < r < R$, where $\|x\|_\infty = \sup_{t \in J} |x(t)|$.

2. Preliminaries

In order to define the solution of problem (1.1), we shall consider the following space.

Let $J' = J \setminus \{t_1, t_2, \ldots, t_n\}$, and

$$PC^1[0,1] = \{x \in C[0,1] : x'|_{(t_k,t_{k+1})} \in C(t_k, t_{k+1}),\ x'(t_k^-) = x'(t_k),\ \exists\, x'(t_k^+),\ k = 1, 2, \ldots, m\}.$$

Then $PC^1[0,1]$ is a real Banach space with norm

$$\|x\|_{pc^1} = \max\{\|x\|_\infty, \|x'\|_\infty\},$$

where $\|x'\|_\infty = \sup_{t\in J} |x'(t)|$.

A function $x \in PC^1[0,1] \cap C^2(J')$ is called a solution of problem (1.1) if it satisfies (1.1).

To establish the existence of multiple positive solutions in $PC^1[0,1] \cap C^2(J')$ of problem (1.1), let us list the following assumptions:

(H_1) $f \in C(J \times R^+, R^+)$, $\quad I_k \in C(R^+, R^+)$;
(H_2) $\Delta \neq 0$,

where

$$\Delta = \begin{vmatrix} -\sum_{i=1}^{m-2} a_i\xi_i & 1 - \sum_{i=1}^{m-2} a_i(1-\xi_i) \\ 1 - \sum_{i=1}^{m-2} b_i\xi_i & -\sum_{i=1}^{m-2} b_i(1-\xi_i) \end{vmatrix}. \tag{2.1}$$

Lemma 2.1. *Assume that* (H_1) *and* (H_2) *hold. Then $x \in PC^1[0,1] \cap C^2(J')$ is a solution of problem* (1.1) *if and only if x is a solution of the following impulsive integral equation:*

$$\begin{aligned} x(t) = {} & \int_0^1 G(t,s) f(s, x(s))\mathrm{d}s + \sum_{k=1}^{n} G(t, t_k) I_k(x(t_k)) \\ & + t[A(f(\cdot, x(\cdot))) + B(I_k(x(\cdot)))] + (1-t)[C(f(\cdot, x(\cdot))) + D(I_k(x(\cdot)))], \end{aligned} \tag{2.2}$$

where

$$G(t,s) = \begin{cases} s(1-t), & \text{if } 0 \le s \le t \le 1, \\ t(1-s), & \text{if } 0 \le t \le s \le 1, \end{cases} \tag{2.3}$$

$$A(f(\cdot, x(\cdot))) := \frac{1}{\Delta} \begin{vmatrix} \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, t) f(t, x(t))\mathrm{d}t & 1 - \sum_{i=1}^{m-2} a_i(1-\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i, t) f(t, x(t))\mathrm{d}t & -\sum_{i=1}^{m-2} b_i(1-\xi_i) \end{vmatrix}, \tag{2.4}$$

$$B(I_k(x(\cdot))) := \frac{1}{\Delta} \begin{vmatrix} \sum_{i=1}^{m-2} a_i \left(\sum_{k=1}^{n} G(\xi_i, t_k) I_k(x(t_k)) \right) & 1 - \sum_{i=1}^{m-2} a_i(1-\xi_i) \\ \sum_{i=1}^{m-2} b_i \left(\sum_{k=1}^{n} G(\xi_i, t_k) I_k(x(t_k)) \right) & -\sum_{i=1}^{m-2} b_i(1-\xi_i) \end{vmatrix}, \tag{2.5}$$

$$C(f(\cdot, x(\cdot))) := \frac{1}{\Delta} \begin{vmatrix} -\sum_{i=1}^{m-2} a_i\xi_i & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, t) f(t, x(t))\mathrm{d}t \\ 1 - \sum_{i=1}^{m-2} b_i\xi_i & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i, t) f(t, x(t))\mathrm{d}t \end{vmatrix}, \tag{2.6}$$

$$D(I_k(x(\cdot))) := \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\xi_i & \sum_{i=1}^{m-2} a_i\left(\sum_{k=1}^{n} G(\xi_i, t_k)I_k(x(t_k))\right) \\ 1-\sum_{i=1}^{m-2} b_i\xi_i & \sum_{i=1}^{m-2} b_i\left(\sum_{k=1}^{n} G(\xi_i, t_k)I_k(x(t_k))\right) \end{vmatrix}. \tag{2.7}$$

Proof. First suppose that $x \in PC^1[0,1] \cap C^2(J')$ is a solution of problem (1.1). It is easy to see by integration of (1.1) that

$$\begin{aligned} x'(t) &= x'(0) - \int_0^t f(s, x(s))\mathrm{d}s + \sum_{0<t_k<t}[x'(t_k^+) - x'(t_k)] \\ &= x'(0) - \int_0^t f(s, x(s))\mathrm{d}s - \sum_{0<t_k<t} I_k(x(t_k)). \end{aligned}$$

Integrate again, we can get

$$x(t) = x(0) + x'(0)t - \int_0^t (t-s)f(s, x(s))\mathrm{d}s - \sum_{0<t_k<t} I_k(x(t_k))(t-t_k). \tag{2.8}$$

Letting $t = 1$ in (2.8), we find

$$x'(0) = \sum_{i=1}^{m-2} b_i x(\xi_i) - \sum_{i=1}^{m-2} a_i x(\xi_i) + \int_0^1 (1-s)f(s, x(s))\mathrm{d}s + \sum_{k=1}^{n} I_k(x(t_k))(1-t_k). \tag{2.9}$$

Substituting $x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i)$ and (2.9) into (2.8), we obtain

$$\begin{aligned} x(t) &= \sum_{i=1}^{m-2} a_i x(\xi_i) + t\left[\sum_{i=1}^{m-2} b_i x(\xi_i) - \sum_{i=1}^{m-2} a_i x(\xi_i) + \int_0^1 (1-s)f(s, x(s))\mathrm{d}s \right. \\ &\quad \left. + \sum_{k=1}^{n} I_k(x(t_k))(1-t_k)\right] - \int_0^t (t-s)f(s, x(s))\mathrm{d}s - \sum_{0<t_k<t} I_k(x(t_k))(t-t_k) \\ &= \int_0^1 G(t,s)f(s, x(s))\mathrm{d}s + \sum_{k=1}^{n} G(t, t_k)I_k(x(t_k)) + t\sum_{i=1}^{m-2} b_i x(\xi_i) + (1-t)\sum_{i=1}^{m-2} a_i x(\xi_i). \end{aligned} \tag{2.10}$$

Then, we have

$$x(\xi_i) = \int_0^1 G(\xi_i, s)f(s, x(s))\mathrm{d}s + \sum_{k=1}^{n} G(\xi_i, t_k)I_k(x(t_k)) + \xi_i\sum_{i=1}^{m-2} b_i x(\xi_i) + (1-\xi_i)\sum_{i=1}^{m-2} a_i x(\xi_i); \tag{2.11}$$

$$\begin{aligned} \sum_{i=1}^{m-2} a_i x(\xi_i) &= \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, s)f(s, x(s))\mathrm{d}s + \sum_{i=1}^{m-2} a_i \sum_{k=1}^{n} G(\xi_i, t_k)I_k(x(t_k)) \\ &\quad + \sum_{i=1}^{m-2} a_i\xi_i \sum_{i=1}^{m-2} b_i x(\xi_i) + \sum_{i=1}^{m-2} a_i(1-\xi_i)\sum_{i=1}^{m-2} a_i x(\xi_i); \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \sum_{i=1}^{m-2} b_i x(\xi_i) &= \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i, s)f(s, x(s))\mathrm{d}s + \sum_{i=1}^{m-2} b_i \sum_{k=1}^{n} G(\xi_i, t_k)I_k(x(t_k)) \\ &\quad + \sum_{i=1}^{m-2} b_i\xi_i \sum_{i=1}^{m-2} b_i x(\xi_i) + \sum_{i=1}^{m-2} b_i(1-\xi_i)\sum_{i=1}^{m-2} a_i x(\xi_i). \end{aligned} \tag{2.13}$$

So, from (2.12) and (2.13), we have

$$\left(1-\sum_{i=1}^{m-2}a_i(1-\xi_i)\right)\sum_{i=1}^{m-2}a_ix(\xi_i)-\sum_{i=1}^{m-2}a_i\xi_i\sum_{i=1}^{m-2}b_ix(\xi_i)$$
$$=\sum_{i=1}^{m-2}a_i\int_0^1G(\xi_i,s)f(s,x(s))\mathrm{d}s+\sum_{i=1}^{m-2}a_i\sum_{k=1}^{n}G(\xi_i,t_k)I_k(x(t_k)); \tag{2.14}$$

$$-\sum_{i=1}^{m-2}b_i(1-\xi_i)\sum_{i=1}^{m-2}a_ix(\xi_i)+\left(1-\sum_{i=1}^{m-2}b_i(1-\xi_i)\right)\sum_{i=1}^{m-2}b_ix(\xi_i)$$
$$=\sum_{i=1}^{m-2}b_i\int_0^1G(\xi_i,s)f(s,x(s))\mathrm{d}s+\sum_{i=1}^{m-2}b_i\sum_{k=1}^{n}G(\xi_i,t_k)I_k(x(t_k)). \tag{2.15}$$

Let $\sum_{i=1}^{m-2}a_ix(\xi_i)=C(f(\cdot,x(\cdot)))+D(I_k(x(\cdot)))$, $\sum_{i=1}^{m-2}b_ix(\xi_i)=A(f(\cdot,x(\cdot)))+B(I_k(x(\cdot)))$. Then, the proof of sufficient is complete.

Conversely, if x is a solution of (2.2). Direct differentiation of (2.2) implies, for $t\neq t_k$

$$x'(t)=-\int_0^t sf(s,x(s))\mathrm{d}s+\int_t^1(1-s)f(s,x(s))\mathrm{d}s-\sum_{k=1}^{n}t_kI_k(x(t_k))$$
$$+\sum_{k=1}^{n}(1-t_k)I_k(x(t_k))+A(f(\cdot,x(\cdot)))+B(I_k(x(\cdot)))-C(f(\cdot,x(\cdot)))-D(I_k(x(\cdot))). \tag{2.16}$$

Evidently,

$$\Delta x'|_{t=t_k}=-I_k(x(t_k)),\quad (k=1,2,\ldots,m),$$

and

$$x''(t)=-f(t,x(t)). \tag{2.17}$$

So $x\in C^2(J')$ and $\Delta x'|_{t=t_k}=-I_k(x(t_k))$, $(k=1,2,\ldots,m)$, and it is easy to verify that $x(0)=\sum_{i=1}^{m-2}a_ix(\xi_i)$, $x(1)=\sum_{i=1}^{m-2}b_ix(\xi_i)$, and the lemma is proved. $\diamond$

Lemma 2.2. *Let* (H_1) *hold. Assume that*
(H_3) $\Delta<0$, $\sum_{i=1}^{m-2}b_i\xi_i<1$, $\sum_{i=1}^{m-2}a_i(1-\xi_i)<1$.
Then, the solution x of problem (1.1) satisfies $x(t)\geq 0$, for $t\in J$.

Proof. It is an immediate subsequence of the facts that $G\geq 0$ on $[0,1]\times[0,1]$ and $A(f(\cdot,x(\cdot)))\geq 0$, $C(f(\cdot,x(\cdot)))\geq 0$, $B(I_k(x(\cdot)))\geq 0$, $D(I_k(x(\cdot)))\geq 0$. $\diamond$

Remark 2. From (2.3), one can find that

$$t_1(1-t_n)G(s,s)\leq G(t,s)\leq G(s,s),\quad t\in[t_1,t_n],s\in J,$$

and

$$G(t,s)\geq t_1(1-t_n),\quad t,s\in[t_1,t_n].$$

For the sake of applying Lemma 1.1, we construct a cone K in $PC^1[0,1]$ by

$$K=\{x\in PC^1[0,1]:x\geq 0,x(t)\geq t_1(1-t_n)x(s),\ t\in[t_1,t_n],s\in J\}. \tag{2.18}$$

Define $T:K\to K$ by

$$(Tx)(t)=\int_0^1G(t,s)f(s,x(s))\mathrm{d}s+\sum_{k=1}^{n}G(t,t_k)I_k(x(t_k))$$
$$+t[A(f(\cdot,x(\cdot)))+B(I_k(x(\cdot)))]+(1-t)[C(f(\cdot,x(\cdot)))+D(I_k(x(\cdot)))]. \tag{2.19}$$

Lemma 2.3. *Assume that* (H$_1$) *and* (H$_3$) *hold. Then,* $T(K) \subset K$ *and* $T : K \to K$ *is completely continuous.*

Proof. For $x \in K$, by (2.19), Lemmas 2.1 and 2.2, we have $Tx \geq 0,\ Tx \in PC^1[0, 1]$, and

$$\begin{aligned}(Tx)(t) \leq& \int_0^1 G(s,s)f(s,x(s))\mathrm{d}s + \sum_{k=1}^n G(t_k,t_k)I_k(x(t_k)) \\ &+ t[A(f(\cdot,x(\cdot))) + B(I_k(x(\cdot)))] + (1-t)[C(f(\cdot,x(\cdot))) + D(I_k(x(\cdot)))], \quad t \in J.\end{aligned}$$

On the other hand, by $t_1(1-t_n) \leq 1$, Remark 2, (2.19), we have

$$\begin{aligned}(Tx)(t) \geq& \ t_1(1-t_n)\int_0^1 G(s,s)f(s,x(s))\mathrm{d}s + t_1(1-t_n)\sum_{k=1}^n G(t_k,t_k)I_k(x(t_k)) \\ &+ t_1(1-t_n)\left\{t[A(f(\cdot,x(\cdot))) + B(I_k(x(\cdot)))] + (1-t)[C(f(\cdot,x(\cdot))) + D(I_k(x(\cdot)))]\right\} \\ =& \ t_1(1-t_n)\left\{\int_0^1 G(s,s)f(s,x(s))\mathrm{d}s + \sum_{k=1}^n G(t_k,t_k)I_k(x(t_k))\right. \\ &\left. +\ t[A(f(\cdot,x(\cdot))) + B(I_k(x(\cdot)))] + (1-t)[C(f(\cdot,x(\cdot))) + D(I_k(x(\cdot)))]\right\} \\ \geq& \ t_1(1-t_n)\|Tx\|_{pc} \geq t_1(1-t_n)(Tx)(u), \quad t \in [t_1,t_n], u \in J.\end{aligned}$$

Thus, $T(K) \subset K$. Next, by similar arguments in [8] one can prove that $T : K \to K$ is completely continuous. So it is omitted, and the lemma is proved. ◇

3. Main results

Write

$$f^\beta = \limsup_{x\to\beta}\max_{t\in J}\frac{f(t,x)}{x}, \qquad f_\beta = \liminf_{x\to\beta}\min_{t\in J}\frac{f(t,x)}{x},$$
$$I_\beta(k) = \liminf_{x\to\beta}\frac{I_k(x)}{x}, \qquad I^\beta(k) = \limsup_{x\to\beta}\frac{I_k(x)}{x},$$

where β denotes 0^+ or $+\infty$.

In this section, we apply Lemma 1.1 to establish the existence of positive solutions for BVP (1.1).

Theorem 3.1. *Assume that* (H$_1$) *and* (H$_3$) *hold. In addition, letting* f *and* I_k *satisfy the following conditions*

(H$_4$) $f^0 = 0$ *and* $I^0(k) = 0, k = 1, 2, \ldots, n$;

(H$_5$) $f_\infty = \infty$ *or* $I_\infty(k) = \infty, k = 1, 2, \ldots, n$.

Then BVP (1.1) *has at least one positive solution.*

Proof. Let T be a cone preserving, completely continuous operator that was defined by (2.19).

Considering (H$_4$), there exists $0 < r < \eta$ such that $f(t,x) \leq \varepsilon r, I_k(x) \leq \varepsilon_k r, k = 1, 2, \ldots, n$, for $0 \leq x \leq r, t \in J$, where $\varepsilon, \varepsilon_k > 0$ satisfy $\varepsilon + \sum_{k=1}^n \varepsilon_k + \tilde{A} + \tilde{B} + \tilde{C} + \tilde{D} < 1$, where

$$\tilde{A} := \frac{1}{\Delta}\begin{vmatrix}\sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i,t)\varepsilon\mathrm{d}t & 1-\sum_{i=1}^{m-2} a_i(1-\xi_i) \\ \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i,t)\varepsilon\mathrm{d}t & -\sum_{i=1}^{m-2} b_i(1-\xi_i)\end{vmatrix},$$

$$\tilde{B} := \frac{1}{\Delta}\begin{vmatrix}\sum_{i=1}^{m-2} a_i \left(\sum_{k=1}^n G(\xi_i,t_k)\varepsilon_k\right) & 1-\sum_{i=1}^{m-2} a_i(1-\xi_i) \\ \sum_{i=1}^{m-2} b_i \left(\sum_{k=1}^n G(\xi_i,t_k)\varepsilon_k\right) & -\sum_{i=1}^{m-2} b_i(1-\xi_i)\end{vmatrix},$$

$$\tilde{C} := \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\xi_i & \sum_{i=1}^{m-2} a_i \int_0^1 G(\xi_i, t)\varepsilon \mathrm{d}t \\ 1-\sum_{i=1}^{m-2} b_i\xi_i & \sum_{i=1}^{m-2} b_i \int_0^1 G(\xi_i, t)\varepsilon \mathrm{d}t, \end{vmatrix},$$

$$\tilde{D} := \frac{1}{\Delta}\begin{vmatrix} -\sum_{i=1}^{m-2} a_i\xi_i & \sum_{i=1}^{m-2} a_i \left(\sum_{k=1}^{n} G(\xi_i, t_k)\varepsilon_k\right) \\ 1-\sum_{i=1}^{m-2} b_i\xi_i & \sum_{i=1}^{m-2} b_i \left(\sum_{k=1}^{n} G(\xi_i, t_k)\varepsilon_k\right) \end{vmatrix}.$$

Now, we prove that

$$Tx \not\geq x, \quad x \in K, \qquad \|x\|_{pc^1} = r. \tag{3.1}$$

In fact, if there exists $x_1 \in K$, $\|x_1\|_{pc^1} = r$ such that $Tx_1 \geq x_1$, then we have

$$\begin{aligned} 0 \leq x_1(t) \ &\leq \ \int_0^1 G(t,s)f(s,x_1(s))\mathrm{d}s + \sum_{k=1}^{n} G(t,t_k)I_k(x_1(t_k)) \\ &\quad + t[A(f(\cdot,x_1(\cdot))) + B(I_k(x_1(\cdot)))] + (1-t)[C(f(\cdot,x_1(\cdot))) + D(I_k(x(\cdot)))] \\ &\leq \ \frac{1}{4}r\varepsilon + \frac{1}{4}r\sum_{k=1}^{n}\varepsilon_k + r\tilde{A} + r\tilde{B} + r\tilde{C} + r\tilde{D} \\ &= \ r\left[\frac{1}{4}\varepsilon + \frac{1}{4}\sum_{k=1}^{n}\varepsilon_k + \tilde{A} + \tilde{B} + \tilde{C} + \tilde{D}\right] \\ &< \ r = \|x_1\|_{pc^1}, \\ |x_1'(t)| \ &\leq \ \int_0^1 |G_t'(t,s)|f(s,x_1(s))\mathrm{d}s + \sum_{k=1}^{n} |G_t'|(t,t_k)I_k(x_1(t_k)) \\ &\quad + A(f(\cdot,x_1(\cdot))) + B(I_k(x_1(\cdot))) + [C(f(\cdot,x_1(\cdot))) + D(I_k(x(\cdot)))] \\ &\leq \ r\varepsilon + r\sum_{k=1}^{n}\varepsilon_k + r\tilde{A} + r\tilde{B} + r\tilde{C} + r\tilde{D} \\ &= \ r\left[\varepsilon + \sum_{k=1}^{n}\varepsilon_k + \tilde{A} + \tilde{B} + \tilde{C} + \tilde{D}\right] \\ &< \ r = \|x_1\|_{pc^1}, \end{aligned}$$

where

$$G_t'(t,s) = \begin{cases} -s, & \text{if } 0 \leq s \leq t \leq 1, \\ 1-s, & \text{if } 0 \leq t \leq s \leq 1, \end{cases} \tag{3.2}$$

and

$$\max_{t,s\in J, t\neq s} |G_t'(t,s)| = 1.$$

Therefore, $\|x_1\|_{pc^1} < \|x_1\|_{pc^1}$, which is a contraction. Hence, (3.1) holds.

Next, turning to (H_5). Case (1), $f_\infty = \infty$. There exists $\tau > 0$ such that

$$f(t,x) \geq Mx, \quad t \in J, \ x \geq \tau,$$

where $M > [t_1(1-t_n)(t_n - t_1)]^{-1}$. Choose

$$R > \max\{r, \tau[t_1(1-t_n)]^{-1}\}. \tag{3.3}$$

We show that

$$Tx \not\leq x, \quad x \in K, \|x\|_{pc^1} = R. \tag{3.4}$$

In fact, if there exists $x_0 \in K,\ \|x_0\|_{pc^1} = R$ such that $Tx_0 \leq x_0$, then

$$x_0(t) \geq t_1(1-t_n)x_0(s), \quad t \in [t_1, t_n], s \in J.$$

This and (3.3) imply that

$$\min_{t\in[t_1,t_n]} x_0(t) \geq t_1(1-t_n)\|x_0\|_{pc^1} = t_1(1-t_n)R > \tau. \tag{3.5}$$

So, we have

$$t \in J \Longrightarrow x_0(t) \geq Tx_0(t) \geq \min_{t\in[t_1,t_n]} \int_{t_1}^{t_n} G(t,s)f(s,x_0(s))\mathrm{d}s \geq t_1(1-t_n)M\int_{t_1}^{t_n} x_0(s)\mathrm{d}s,$$

i.e.,

$$\int_{t_1}^{t_n} x_0(t)\mathrm{d}t \geq t_1(1-t_n)M(t_n-t_1)\int_{t_1}^{t_n} x_0(s)\mathrm{d}s. \tag{3.6}$$

It is easy to see that

$$\int_{t_1}^{t_n} x_0(s)\mathrm{d}s > 0. \tag{3.7}$$

In fact, if $\int_{t_1}^{t_n} x_0(s)\mathrm{d}s = 0$, then $x_0(t) = 0$, for $t \in [t_1, t_n]$. Since $x_0 \in K,\ x_0(s) = 0$ for any $s \in J$. Hence, $\|x_0\|_{pc^1} = \|x_0'\|_\infty = \|x_0\|_\infty = 0$, which contracts $\|x_0\|_{pc^1} = R$. So, (3.7) holds. Therefore, $M \leq [t_1(1-t_n)(t_n-t_1)]^{-1}$, this is also a contraction. Hence, (3.4) holds.

Case (2), $I_\infty(k) = \infty,\ \ k = 1, 2, \ldots, n$. There exists $\tau_1 > 0$ such that

$$I_k(x) \geq M_k x, \quad x \geq \tau_1,$$

where $M_k > [t_1(1-t_n)]^{-1}$, $k = 1, 2, \ldots, n$. If we define $M^* = \min\{M_k : k = 1, 2, \ldots, n\}$, then $M^* > [t_1(1-t_n)]^{-1}$. Choose

$$R > \max\{r, \tau_1[t_1(1-t_n)]^{-1}\}. \tag{3.8}$$

We prove that (3.4) holds.

In fact, if there exists $x_{00} \in K, \|x_{00}\|_{pc^1} = R$ such that $Tx_{00} \leq x_{00}$, then

$$x_{00}(t) \geq t_1(1-t_n)x_{00}(s), \quad t \in [t_1, t_n], s \in J.$$

This and (3.8) imply that

$$\min_{t\in[t_1,t_n]} x_{00}(t) \geq t_1(1-t_n)\|x_{00}\|_{pc^1} = t_1(1-t_n)R > \tau_1. \tag{3.9}$$

So, we have

$$\begin{aligned} t \in J \ \Longrightarrow\ x_{00}(t) \geq Tx_{00}(t) &\geq \min_{t\in[t_1,t_n]} \sum_{k=1}^{n} G(t,t_k)I_k(x_{00}(t_k)) \\ &\geq t_1(1-t_n)\sum_{k=1}^{n} M_k x_{00}(t_k) \\ &\geq t_1(1-t_n)M^*\sum_{k=1}^{n} x_{00}(t_k). \end{aligned} \tag{3.10}$$

From (3.10), we obtain that

$$x_{00}(t_1) \geq t_1(1-t_n)M^* \sum_{k=1}^{n} x_{00}(t_k),$$
$$x_{00}(t_2) \geq t_1(1-t_n)M^* \sum_{k=1}^{n} x_{00}(t_k),$$
$$\ldots,$$
$$x_{00}(t_k) \geq t_1(1-t_n)M^* \sum_{k=1}^{n} x_{00}(t_k).$$

So, we have

$$\sum_{k=1}^{n} x_{00}(t_k) \geq nt_1(1-t_n)M^* \sum_{k=1}^{n} x_{00}(t_k).$$

From the definition of M^*, we can find that

$$\sum_{k=1}^{n} x_{00}(t_k) > n \sum_{k=1}^{n} x_{00}(t_k), \quad x_{00} \in K,\ \|x_{00}\|_{pc^1} = R. \tag{3.11}$$

Similar to the proof of that in case (1), we can show that $\sum_{k=1}^{n} x_{00}(t_k) > 0$. Then, from (3.11), we have $n < 1$, which is a contraction. Hence, (3.4) holds.

Applying (i) of Lemma 1.1 to (3.1) and (3.4) yields that T has a fixed point $x \in \bar{K}_{r,R} = \{x : r \leq \|x^*\|_{pc^1} \leq R\}$. Thus, it follows that BVP (1.1) has one positive solution, and the theorem is proved. ◇

Theorem 3.2. *Assume that* (H_1) *and* (H_3) *hold. In addition, letting* f *and* I_k *satisfy the following conditions*

(H_6) $f^{\infty} = 0$ *and* $I^{\infty}(k) = 0$, $k = 1, 2, \ldots, n$;
(H_7) $f_0 = \infty$ *or* $I_0(k) = \infty$, $k = 1, 2, \ldots, n$.

Then BVP (1.1) *has at least one positive solution.*

Proof. Let T be cone preserving, completely continuous operator that was defined by (2.19).

Considering (H_6), there exists $\bar{r} > 0$ such that $f(t, x) \leq \bar{\varepsilon}\bar{r}$, $I_k(x) \leq \bar{\varepsilon}_k\bar{r}$, $k = 1, 2, \ldots, n$, for $x \geq \bar{r}$, $t \in J$, where $\bar{\varepsilon}, \bar{\varepsilon}_k > 0$ satisfy $\bar{\varepsilon} + \sum_{k=1}^{n} \bar{\varepsilon}_k + \tilde{A} + \tilde{B} + \tilde{C} + \tilde{D} < 1$.

Similar to the proof of (3.1), we can show that

$$Tx \ngeq x, \quad x \in K, \|x\|_{pc^1} = \bar{r}. \tag{3.12}$$

Next, turning to (H_7). Under the condition (H_7), similar to the proof of (3.4), we can also show that

$$Tx \nleq x, \quad x \in K, \|x\|_{pc^1} = \bar{R}. \tag{3.13}$$

Applying (i) of Lemma 1.1 to (3.12) and (3.13) yields that T has a fixed point $\bar{x} \in \bar{K}_{\bar{r},\bar{R}} = \{x : \bar{r} \leq \|x\|_{pc^1} \leq \bar{R}\}$. Thus, it follows that BVP (1.1) has one positive solution, and the theorem is proved. ◇

Theorem 3.3. *Assume that* (H_1), (H_3), (H_4) *and* (H_6) *hold. In addition, letting* f *and* I_k *satisfy the following condition*

(H_8) *There is a* $\eta > 0$ *such that* $t_1(1-t_n)\eta \leq x \leq \eta$ *and* $t \in J$ *implies*

$$f(t, x) \geq \tau\eta, \quad I_k(x) \geq \tau_k\eta,$$

where $\bar{\varepsilon}, \bar{\varepsilon}_k \geq 0$ *satisfy* $\bar{\varepsilon} + \sum_{k=1}^{n} \bar{\varepsilon}_k > 0$, $\bar{\varepsilon} \int_{t_1}^{t_m} G(\frac{1}{2}, s)\mathrm{d}s + \sum_{k=1}^{n} \varepsilon_k G(\frac{1}{2}, t_k) > 1$. *Then BVP* (1.1) *has at least two positive solutions* x^* *and* x^{**} *with* $0 < \|x^*\|_{pc^1} < \eta < \|x^{**}\|_{pc^1}$.

Proof. Choose $0 < \rho < \eta < \gamma$. If (H_4) holds, similar to the proof of (3.1), we can prove that

$$Tx \ngeq x, \quad x \in K, \|x\|_{pc^1} = \rho. \tag{3.14}$$

If (H_6) holds, similar to the proof of (3.12), we have

$$Tx \ngeq x, \quad x \in K, \|x\|_{pc^1} = \gamma. \tag{3.15}$$

Finally, we show that

$$Tx \nleq x, \quad x \in K, \|x\|_{pc^1} = \eta. \tag{3.16}$$

In fact, if there exists $x_2 \in K$ with $\|x_2\|_{pc^1} = \eta$, then by (2.18), we have

$$x_2(t) \geq t_1(1-t_n)\|x_2\|_{pc^1} = t_1(1-t_n)\{t, 1-t\}\eta$$

and it follows from (H_8) that

$$\begin{aligned} x_2(t) &\geq \int_{t_1}^{t_n} G\left(\frac{1}{2}, s\right) f(s, x_2(s))\mathrm{d}s + \sum_{k=1}^{n} G\left(\frac{1}{2}, t_k\right) I_k(x_2(t_k)) \\ &\geq \eta\left[\bar{\varepsilon}\int_{t_1}^{t_m} G\left(\frac{1}{2}, s\right)\mathrm{d}s + \sum_{k=1}^{n} \varepsilon_k G\left(\frac{1}{2}, t_k\right)\right] \\ &> \eta = \|x_2\|_{pc^1}, \end{aligned} \tag{3.17}$$

i.e., $\|x_2\|_{pc^1} > \|x_2\|_{pc^1}$, which is a contraction. Hence, (3.16) holds.

Applying Lemma 1.1 to (3.14)–(3.16) yields that T has two fixed points x^*, x^{**} with $x^* \in \bar{K}_{\rho,\eta}, x^{**} \in \bar{K}_{\eta,\gamma}$. Thus it follows that BVP (1.1) has two positive solutions x^*, x^{**} with $0 < \|x^*\|_{pc^1} < \eta < \|x^{**}\|_{pc^1}$. The proof is complete. ◇

4. Example

To illustrate how our main results can be used in practice we present an example.

Example 4.1. Consider the following boundary value problem

$$\begin{cases} -x'' = \sqrt[3]{t^3+1}x^3 \tanh x, \quad t \in J, t \neq \dfrac{1}{2}, \\ -\Delta x'|_{t_1=\frac{1}{2}} = x^2\left(\dfrac{1}{2}\right), \\ x(0) = x(1) = \dfrac{1}{3}x\left(\dfrac{1}{3}\right). \end{cases} \tag{4.1}$$

Conclusion. BVP (4.1) has at least one positive solution $x^*(t)$.

Proof. BVP (4.1) can be regarded as a BVP of the form (1.1), where $a_1 = b_1 = \xi_1 = \frac{1}{3}, t_1 = \frac{1}{2}, f(t,x) = \sqrt[3]{t^3+1}x^3 \tanh x$, $I_1(x) = x^2$. It is not difficult to see that the conditions (H_1) and (H_3) hold. In addition,

$$f^0 = \limsup_{x\to 0} \max_{t\in J} \frac{f(t,x)}{x} = 0, \qquad I^0(k) = \limsup_{x\to 0} \frac{I_k(x)}{x} = 0, \tag{4.2}$$

and

$$f_\infty = \liminf_{x\to\infty} \min_{t\in J} \frac{f(t,x)}{x} = \infty. \tag{4.3}$$

Then, the conditions (H_4) and (H_5) of Theorem 3.1 hold. Hence, by Theorem 3.1, the conclusion follows, and the proof is complete. ◇

Remark 3. Example 4.1 implies that there is a large number of functions that satisfy the conditions of Theorem 3.1. In addition, the conditions of Theorem 3.1 are also easy to check.

Acknowledgement

The authors are indebted to the referee's suggestions that have greatly improved this paper.

References

[1] V. Lakshmikantham, D. Bainov, P. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.
[2] D. Bainov, P. Simeonov, Systems with Impulse Effect, Ellis Horwood, Chichister, 1989.
[3] A.M. Samoilenko, N.A. Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.
[4] M. Feng, H. Pang, A class of three point boundary value problems for second order impulsive integro-differential equations in Banach spaces, Nonlinear Anal. (2007), doi:10.1016/j.na.2007.11.033.
[5] Z. Wei, Periodic boundary value problems for second order impulsive integrodifferential equations of mixed type in Banach spaces, J. Math. Anal. Appl. 195 (1995) 214–229.
[6] S. Hristova, D. Bainov, Monotone-iterative techniques of V. Lakshmikantham for a boundary value problem for systems of impulsive differential-difference equations, J. Math. Anal. Appl. 1997 (1996) 1–13.
[7] X. Liu, D. Guo, Periodic boundary value problems for a class of second-order impulsive integro-differential equations in Banach spaces, Appl. Math. Comput. 216 (1997) 284–302.
[8] R.P. Agarwal, D. O'Regan, Multiple nonnegative solutions for second order impulsive differential equations, Appl. Math. Comput. 114 (2000) 51–59.
[9] B. Liu, J. Yu, Existence of solution for m-point boundary value problems of second-order differential systems with impulses, Appl. Math. Comput. 125 (2002) 155–175.
[10] W. Ding, M. Han, Periodic boundary value problem for the second order impulsive functional differential equations, Appl. Math. Comput. 155A (2004) 709–726.
[11] E. Lee, Y. Lee, Multiple positive solutions of singular two point boundary value problems for second order impulsive differential equation, Appl. Math. Comput. 158 (2004) 745–759.
[12] V. Il'in, E. Moiseev, Nonlocal boundary value problem of the second kind for a Sturm–Liouville operator, Differ. Equ. 23 (1987) 979–987.
[13] R. Ma, Multiplicity of positive solutions for second-order three-point boundary value problems, Comput. Math. Appl. 40 (2000) 193–204.
[14] R. Ma, N. Castaneda, Existence of solutions of nonlinear m-point boundary-value problems, J. Math. Anal. Appl. 256 (2001) 556–567.
[15] R. Ma, H. Wang, Positive solutions of nonlinear three-point boundary-value problems, J. Math. Anal. Appl. 279 (2003) 216–227.
[16] R. Ma, B. Thompson, Positive solutions for nonlinear m-point eigenvalue problems, J. Math. Anal. Appl. 297 (2004) 24–37.
[17] R. Ma, Multiple positive solutions for nonlinear m-point boundary value problems, Appl. Math. Comput. 148 (2004) 249–262.
[18] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, Inc., NewYork, 1988.
[19] Z. Bai, Z. Du, Positive solutions for some second-order four-point boundary value problems, J. Math. Anal. Appl. 330 (2007) 34–50.
[20] Z. Bai, W. Ge, Y. Wang, Multiplicity results for some second-order four-point boundary value problems, Nonlinear Anal. 60 (2005) 491–500.
[21] Z. Bai, W. Li, W. Ge, Existence and multiplicity of solutions for four-point boundary value problems at resonance, Nonlinear Anal. 60 (2005) 1151–1162.
[22] X. He, W. Ge, Triple positive solutions for second-order three-point boundary value problems, J. Math. Anal. Appl. 268 (2002) 256–265.
[23] Y. Guo, W. Ge, Positive solutions for three-point boundary value problems with dependence on the first order derivative, J. Math. Anal. Appl. 290 (2004) 291–301.
[24] W. Cheung, J. Ren, Positive solutions for m-point boundary-value problems, J. Math. Anal. Appl. 303 (2005) 565–575.
[25] C. Gupta, A generalized multi-point boundary value problem for second order ordinary differential equations, Appl. Math. Comput. 89 (1998) 133–146.
[26] W. Feng, On a m-point nonlinear boundary value problems, Nonlinear Anal. 30 (1997) 5369–5370.
[27] W. Feng, J.R.L. Webb, Solvability of a m-point boundary value problems with nonlinear growth, J. Math. Anal. Appl. 212 (1997) 467–480.
[28] W. Feng, J.R.L. Webb, Solvability of a three-point nonlinear boundary value problems at resonance, Nonlinear Anal. 30 (1997) 3227–3238.
[29] Z. Zhang, J. Wang, The upper and lower solution method for a class of singular nonlinear second order three-point boundary value problems, J. Comput. Appl. Math. 147 (2002) 41–52.
[30] G. Zhang, J. Sun, Positive solutions of m-point boundary value problems, J. Math. Anal. Appl. 291 (2004) 406–418.
[31] M. Feng, W. Ge, Positive solutions for a class of m-point singular boundary value problems, Math. Comput. Modelling 46 (2007) 375–383.